BENDING OF A SECTOR OF A STRONGLY ANISOTROPIC ELASTIC RING

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By using a modification of the method in [1] an asymptotic is constructed for the solution of the first bending boundary-value problem (the deflection and angle of rotation are given on the boundary) for a sector of a cylindrical orthotropic ring under the assumption that the bending stiffness in the circumferential direction is considerably greater than the bending stiffness in the radial direction. Application of the method in [1] is made complicated in this situation by the presence of angular points in the domain, which results in the appearance of boundary layer functions of two different kinds in the asymptotic: one described by ordinary differential equations along the characteristic part of the boundary, and the other described by partial differential equations and concentrated near the angular points. The single boundary-value problem for partial differential equations known to the author was studied in [2]. The situation in [2] is simpler than the situation in this paper since the order of the equation is reduced in [2] for $\varepsilon = 0$.

1. Let Q be a domain of the following kind: $Q = \{(r, \theta),\ 0 < a \leqslant r \leqslant b,\ 0 \leqslant \theta \leqslant c\}$. Let us introduce the dimensionless radial coordinate $x = \ln(r/a)$ and let us put $x_0 = \ln(b/a)$, $\varepsilon^2 = D_{11}D_{22}^{-1}$, $b_{12} = D_{12}D_{11}^{-1}$, $b_{66} = D_{66}D_{11}^{-1}$, $m = 2(b_{12} + 2b_{66})$, where D_{ij} are the plate bending stiffnesses. The bending equation for a cylindrical orthotropic plate under the assumption of validity of the Kirchhoff–Love hypotheses has the form [3]

$$\frac{\partial^4 w^\varepsilon}{\partial\theta^4} + 2\frac{\partial^2 w^\varepsilon}{\partial\theta^2} - \frac{\partial^2 w^\varepsilon}{\partial x^2} + 2\frac{\partial w^\varepsilon}{\partial x} + \varepsilon^2\left[\frac{\partial^4 w^\varepsilon}{\partial x^4} - 4\frac{\partial^3 w^\varepsilon}{\partial x^3} + 5\frac{\partial^2 w^\varepsilon}{\partial x^2} - 2\frac{\partial w^\varepsilon}{\partial x} + m\left(\frac{\partial^4 w^\varepsilon}{\partial x^2\partial\theta^2} - 2\frac{\partial^3 w^\varepsilon}{\partial x\partial\theta^2} + \frac{\partial^2 w^\varepsilon}{\partial\theta^2}\right)\right] = f. \quad (1.1)$$

Let us pose the boundary value problem A_ε for (1.1):

$$w^\varepsilon(x, 0) = f_1(x),\quad w^\varepsilon(x, c) = f_2(x),\quad \frac{\partial w^\varepsilon}{\partial\theta}(x, 0) = f_3(x),\quad \frac{\partial w^\varepsilon}{\partial\theta}(x, c) = f_4(x); \quad (1.2)$$

$$w^\varepsilon(0, \theta) = g_1(\theta),\quad w^\varepsilon(x_0, \theta) = g_2(\theta); \quad (1.3)$$

$$\frac{\partial w^\varepsilon}{\partial x}(0, \theta) = g_3(\theta),\quad \frac{\partial w^\varepsilon}{\partial x}(x_0, \theta) = g_4(\theta). \quad (1.4)$$

We shall henceforth perform the whole examination by using formal power series in ε. It is assumed that $f \in C^\infty(Q \cup \partial Q)$, $g_k(\theta)$, $f_k(\theta) \in C^\infty(\partial Q)$, $k = 1, 2, 3, 4$, ∂Q is the boundary of the domain Q.

We seek the approximate solution of the problem A_ε in the form

$$w^0(x, \theta) = \sum_{n=0}^{\infty} \varepsilon^n w_n(x, \theta). \quad (1.5)$$

Substituting (1.5) into (1.1) and collecting like terms in identical powers of ε, we obtain a recurrently related system of equations

$$L(w_0) = f,\ L(w_1) = 0,\ L(w_n) + M(w_{n-2}) = 0,\ n \geqslant 2, \quad (1.6)$$

where $L(w) = \partial^4 w/\partial\theta^4 + 2\partial^2 w/\partial\theta^2 - \partial^2 w/\partial x^2 + 2\partial w/\partial x$; $M(w)$ is a differential operator in the ε^2 power in (1.1). The limit boundary-value problem A_0 is to solve the equations $L(w_0) = f$ under the boundary conditions (1.2), (1.3), where w^ε should be replaced by w_0. Let us note that the boundary conditions (1.4) are not generally satisfied here since the equations for w_n from the system (1.6) are of second order in the variable x and not the fourth. To compensate the residuals that occur, boundary-layer functions must be constructed near the sides

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$x = 0$, x_0.

Let us construct the boundary-layer functions near the side $x = 0$ (they are constructed analogously near the side $x = x_0$ upon introducing the stretched coordinate $t_1 = (x_0 - x)/\varepsilon$). Near $x = 0$ we introduce the stretched coordinate $t = x/\varepsilon$, and substituting $x = t\varepsilon$ in the homogeneous equation (1.1), we obtain

$$\varepsilon^{-2}\left(\frac{\partial^4 w}{\partial\theta^4} + 2\frac{\partial^2 w}{\partial\theta^2} - \varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} + 2\varepsilon^{-1}\frac{\partial w}{\partial t}\right) + \varepsilon^{-4}\frac{\partial^4 w}{\partial t^4} - 4\varepsilon^{-3}\frac{\partial^3 w}{\partial t^3} + 5\varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} - $$
$$- 2\varepsilon^{-1}\frac{\partial w}{\partial t} + m\frac{\partial^2}{\partial\theta^2}\left(\varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} - 2\varepsilon^{-1}\frac{\partial w}{\partial t} + w\right) = 0. \tag{1.7}$$

We seek the approximate solution of (1.7) in the form

$$w^1(t,\theta) = \varepsilon\sum_{n=0}^{\infty}\varepsilon^n w_{n,0}(t,\theta). \tag{1.8}$$

Substituting (1.8) into (1.7) and collecting terms in coincident powers of ε we obtain the recurrently related system of equations

$$\frac{\partial^4 w_{0,0}}{\partial t^4} - \frac{\partial^2 w_{0,0}}{\partial t^2} = 0, \quad \frac{\partial^4 w_{n,0}}{\partial t^4} - \frac{\partial^2 w_{n,0}}{\partial t^2} = M_n^0\ (w_{n-1,0}, \ldots, w_{0,0}), \quad n \geqslant 1. \tag{1.9}$$

The specific form of the differential operators M_n^0 is not essential henceforth and is easily restored from (1.7). We require that the sum $w^0(x, \theta) + w^1(t, \theta)$ satisfy the boundary conditions (1.4), (1.3) at $x = 0$. We hence obtain that the functions $w_n(x, \theta)$, $w_{n,0}(t, \theta)$ satisfy the following boundary conditions for $x = 0$ for $t = 0$:

$$\frac{\partial w_{0,0}}{\partial t}(0,\theta) = g_3(\theta) - \frac{\partial w_0}{\partial x}(0,\theta), \quad \frac{\partial w_{n,0}}{\partial t}(0,\theta) = -\frac{\partial w_n}{\partial x}(0,\theta),$$
$$w_n(0,\theta) = -w_{n-1,0}(0,\theta), \quad n \geqslant 1. \tag{1.10}$$

For complete definiteness of the function $w_n(x, \theta)$ it is necessary to require that for $n \geq 1$

$$\frac{\partial^k w_n}{\partial\theta^k}(0,\theta) = \frac{\partial^k w_n}{\partial\theta^k}(x_0,\theta) = 0, \quad k = 0, 1. \tag{1.11}$$

As follows from [1] and the boundary conditions (1.10), the functions $w_{n,0}(t, \theta)$ have the form

$$w_{n,0}(t,\theta) = \sum_{j=0}^{n-1}\omega_{nj}(\theta)\, t\ \exp(-t)$$

and are determined uniquely from equations (1.9) under the assumption that $w_{n,0}(+\infty, 0) = 0$. The characteristic equation $\lambda^2(\lambda^2 - 1) = 0$ for the system (1.9) has one root with negative real part; therefore, the degeneration of the problem A_ε into the problem A_0 is regular [1]. Analogously to the above, a boundary layer function $w_{n,1}(t_1, \theta)$ can be constructed near the side $x = x_0$. Multiplying $w_{n,0}(t, \theta)$ and $w_{n,1}(t_1, \theta)$ by the truncating functions η and η_1 [1], which are noticeably different from zero only near the sides $x = 0$ and $x = x_0$, respectively, we obtain an asymptotic expansion of the problem A_ε in the form

$$w^\varepsilon(x,\theta) = \sum_{n=0}^{\infty}\varepsilon^n w_n(x,\theta) + \varepsilon\sum_{n=0}^{\infty}\varepsilon^n[\eta w_{n,0}(t,\theta) + \eta_1 w_{n,1}(t_1,\theta)]. \tag{1.12}$$

However, we note that the representation (1.12) of the solution of the problem A_ε inserts a residual in the satisfaction of the boundary conditions for $\theta = 0$, c. Indeed, for $\theta = 0$ the equality

$$\sum_{n=0}^{\infty}\varepsilon^n w_n(x,0) + \varepsilon\sum_{n=0}^{\infty}\varepsilon^n[\eta w_{n,0}(t,0) + \eta_1 w_{n,1}(t_1,0)] = f_1(x)$$

should hold, and we consequently have for $n \geqslant 1$

$$w_n(x, 0) = -\eta w_{n-1,0}(t, 0) - \eta_1 w_{n-1,1}(t_1, 0),$$

which contradicts (1.11) since the boundary conditions for the functions $w_n(x, \theta)$ should not depend on ε.

2. Let us construct an angular boundary-layer function near the point (0, 0) (they are constructed analogously near other points).

Near the point (0, 0) we introduce the stretched coordinates $t = x/\varepsilon$, $\tau = \theta/\sqrt{\varepsilon}$ and by substituting $x = t\varepsilon$, $\theta = \tau\sqrt{\varepsilon}$ in the homogeneous equation (1.1) we obtain

$$\varepsilon^{-2}\left(\varepsilon^{-2}\frac{\partial^4 w}{\partial\tau^4} + 2\varepsilon^{-1}\frac{\partial^2 w}{\partial\tau^2} - \varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} + 2\varepsilon^{-1}\frac{\partial w}{\partial t}\right) + \varepsilon^{-4}\frac{\partial^4 w}{\partial t^4} - \\ - 4\varepsilon^{-3}\frac{\partial^3 w}{\partial t^3} + 5\varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} + m\varepsilon^{-1}\frac{\partial^2}{\partial\tau^2}\left(\varepsilon^{-2}\frac{\partial^2 w}{\partial t^2} - 2\varepsilon^{-1}\frac{\partial w}{\partial t} + w\right) = 0. \tag{2.1}$$

We seek the approximate solution of (2.1) in the form

$$w^2(t, \tau) = \varepsilon\sum_{n=0}^{\infty}\varepsilon^{n/2}p_{n,0}(t, \tau). \tag{2.2}$$

Substituting (2.2) into (2.1) and presenting similar terms, we obtain a recurrently related system of equations

$$\partial^4 p_{0,0}/\partial\tau^4 + \partial^4 p_{0,0}/\partial t^4 - \partial^2 p_{0,0}/\partial t^2 = 0, \\ \partial^4 p_{n,0}/\partial\tau^4 + \partial^4 p_{n,0}/\partial t^4 - \partial^2 p_{n,0}/\partial t^2 = Q_n(p_{n-1,0}, \ldots, p_{0,0}),\ n \geqslant 1. \tag{2.3}$$

The differential operators Q_n are easily reproduced from (2.1); their specific form is not essential for the sequel. We require that the sum

$$\sum_{n=0}^{\infty}\varepsilon^n w_n(x, \theta) + \varepsilon\sum_{n=0}^{\infty}\left[\varepsilon^n w_{n,0}(t, \theta) + \varepsilon^{n/2}p_{n,0}(t, \tau)\right] \tag{2.4}$$

satisfy the boundary conditions of the problem A_ε for $\theta = 0$ and $x = 0$. We hence have boundary conditions for the functions $p_{n,o}(t, \tau)$ for $t = 0$ and $\tau = 0$:

$$p_{n,0}(0, \tau) = \frac{\partial p_{n,0}}{\partial t}(0, \tau) = 0,\quad n \geqslant 0, \\ p_{2n,0}(t, 0) = -w_{n,0}(t, 0),\quad p_{2n+1,0}(t, 0) = 0. \\ \frac{\partial p_{2n,0}}{\partial\tau}(t, 0) = 0,\quad \frac{\partial p_{2n+1,0}}{\partial\tau}(t, 0) = -\frac{\partial w_{n,0}}{\partial\theta}(t, 0). \tag{2.5}$$

Since equations (2.3) are elliptic and the boundary-value problem (2.5) is correct, the solutions $p_{n,o}(t, \tau)$, $n = 0, 1 \ldots$ exist and according to [4] the integral

$$\int_K\left\{\sum_{p=0}^{2}\sum_{q=0}^{2-p}(t^2+\tau^2)^{2-p-q}\left|\frac{\partial^{p+q}}{\partial t^p\partial\tau^q}p_{n,0}(t,\tau)\right|^2\right\}dt\,d\tau$$

is finite for them, where K is a quadrant $K = \{(t, \tau);\ t > 0,\ \tau > 0\}$. As $t \to +\infty$ the $p_{n,o}(t, \tau)$ decrease exponentially. Multiplying the third term in (2.4) by a truncating function that is noticeably different from zero only near the apex of the angle (0, 0), we obtain an asymptotic expansion of the problem A_ε near the apex of the angle.

Therefore, the complete asymptotic expansion of the problem A_ε consists of seven series; the series (1.5) describing the fundamental state of stress, two series describing the boundary layer along the characteristic part of the boundary, and four governing the angular boundary layers.

Let us note that for sufficiently high smoothness of the boundary data, the asymptotic expansion of the problem A_ε allows differentiation and permits the construction of asymptotic expansions for the forces, moments, and transverse forces. Let ε_θ denote the circumferential strain. If $\varepsilon_\theta(w_n) \neq 0$, n = 0, 1, then the asymptotic expansions of the moment M_θ and the transverse forces N_r, N_θ start with the power ε^{-2}.

For $\varepsilon_\theta(w_n) = 0$, n = 0, 1 the plate is inextensible in the circumferential direction.

In conclusion, we note that the asymptotic for the problem of bending a symmetrically assembled anisotropic rectangular laminar shell [5] under strictly nonzero steepness of the family of bonding fibers can be constructed completely analogously (in a formal complication of the computations).

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STABILITY OF MAGNETIC SUSPENSION IN A DIRECT-CURRENT MAGNETIC FIELD

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The problem of suspension of a body for a lengthy period of time using permanent magnets has attracted the interest of researchers. A detailed bibliography of studies of this problem, an analysis of the state of the art, and original results have been presented in [1, 2].

The major result achieved has been Ernshaw's theorem, which indicates the instability of such suspension. However this theorem is concerned with steady state situations, and as we will demonstrate below, is inapplicable to dynamic systems.

1. We will consider the configuration of magnets shown in Fig. 1. We will consider motion of an infinitely long rod in the magnetic channel along the axis Ox. The weight of the rod P = mg is compensated by magnets of one sign 1 or 2. Along the channel sides there is a system of permanent magnets of alternating polarity, which interacts with a similar system located on the rod. We will assume that the pole step along the axis Ox is equal to $\lambda = 2\pi/k$, where k is the wave number. We will assume the magnetic material to be saturated with a value of $\mu = 1$ (where μ is the relative permittivity), as in a vacuum. Considering further that magnet system 3 has a vertical length, we will neglect forces produced by interaction of magnets 3 and 4 during vertical oscillations of the rod.

We will now perform some preliminary calculations. At the point $M_0(x_0, y_0, z_0)$ let there be some magnetic charge q. Its potential at the point M(x, y, z) is equal to $U = q/4\pi\mu_0 r$, $r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, where μ_0 is the absolute magnetic permittivity of free space. The force produced by interaction of two charges q^+, q^- located at these points is given by the expression

$$F = \frac{1}{4\pi\mu_0}\frac{q^+q^-}{r^2}$$

and is directed along the vector joining the charges.

We will consider an infinite magnetic pole located along the axis Oz. For an element of the pole dz the magnetic charge is equal to $dq = \gamma^+ dz$, where γ^+ is the linear charge density. The force of interaction with an analogous elementary charge sectioned from another magnetic

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